Measurings. The natural map from an algebra A to to the dual algebra of a coalgebra C

$$A \longrightarrow \mathbf{Vect}(C, \mathbb{C})$$

can be written as

$$a \longmapsto (c \longmapsto a(c)),$$

which is equivalent to a map

$$C \otimes A \longrightarrow \mathbb{C}$$

$$c \otimes a \longmapsto c(a) := a(c)$$

s.t. $\qquad c(a_1 a_2) = c_{(1)}(a_1)\, c_{(2)}(a_2)$.

Such coalgebra actions on an algebra are called measurings (in $\mathbb{C}$). $\mathbb{C}$ can be replaced by an arbitrary algebra B, since $\mathbf{Vect}(C,B)$ has a natural structure of an algebra

$$\mathbf{Vect}(C,B)\otimes\mathbf{Vect}(C,B)\to\mathbf{Vect}(C\otimes C,B)\xrightarrow{\mathbf{Vect}(\Delta,B)}\mathbf{Vect}(C,B),$$

$$\beta_1\otimes\beta_2\longmapsto\big(c_1\otimes c_2\mapsto\beta_1(c_1)\beta_2(c_2)\big)$$

$$\mathbb{C}\longrightarrow\mathbf{Vect}(\mathbb{C},\mathbb{C})\xrightarrow{\mathbf{Vect}(\varepsilon,B)}\mathbf{Vect}(C,\mathbb{C})\xrightarrow{\mathbf{Vect}(C,\eta)}\mathbf{Vect}(C,B).$$

Therefore every algebra B defines a functor

$$\mathbf{Coalg}\longrightarrow\mathbf{Alg}^{op}$$
$$C\rightsquigarrow\mathbf{Vect}(C,B)$$

Theorem. For every algebra A there is an adjunction

$$\mathbf{Coalg}\big(C, M(B,A)\big) = \mathbf{Alg}^{op}\big(\mathbf{Vect}(C,A), B\big).$$

Proof. We will construct a coalgebra $M(B,A)$ measuring B in A

$$\theta : M(B,A) \otimes B \longrightarrow A$$

satisfying the following <u>universal property</u>:

For every measuring of B in A $\quad \psi : C \otimes B \longrightarrow A$ there exists a unique coalgebra map $F : C \longrightarrow M(B,A)$ making the diagram measuring

$$
\begin{tikzcd}
C \otimes B \arrow[r, "\forall \psi"] \arrow[dr, dashed, "\exists ! F \otimes B"'] & A \\
& M(B,A) \otimes B \arrow[u, "\theta"']
\end{tikzcd}
$$

commute.
To this end we will use the following construction of cofree algebras.

Definition. For every vector space V a <u>cofree coalgebra on V</u> is a coalgebra $C(V)$ together with a linear map $\pi : C(V) \longrightarrow V$, if for every linear map $\pi : C \longrightarrow V$ there is a unique coalgebra map $F : C \longrightarrow C(V)$ making the diagram

linear maps

coalgebra map

$$\begin{tikzcd} C \arrow[r, "\forall f"] \arrow[dr, "\exists ! F"'] & V \\ & C(V) \arrow[u, "\pi"'] \end{tikzcd}$$

commute.

Theorem. The cofree coalgebra $C(V) \to V$ always exists.

Proof. Let $T(V^*)$ will be the tensor algebra on V^* and $i : V^* \longrightarrow T(V^*)$ the natural injection.

Let π be the composite

$$T(V^*)^\circ \hookrightarrow T(V^*)^* \xrightarrow{i^*} V^{**}.$$

We claim that $\pi : T(V^*)^\circ \longrightarrow V^{**}$ is a cofree coalgebra on V^{**}.

We have a natural bijection

$$\mathbf{Vect}(X, Y^*) \cong \mathbf{Vect}(Y, X^*)$$

given by

$$f \longmapsto (Y \hookrightarrow Y^{**} \xrightarrow{f^*} X^*)$$

$$(X \hookrightarrow X^{**} \xrightarrow{g^*} Y^*) \longleftarrow\!\shortmid\; g$$

Taking $X = C$, $Y = V^*$ we have

α) $$\mathbf{Vect}(C, V^{**}) \cong \mathbf{Vect}(V^*, C^*)$$

C coalgebra $\Longrightarrow$ C^* algebra and by universal property (freeness) of $T(V^*)$ we have

β) $$\mathbf{Vect}(V^*, C^*) \cong \mathbf{Alg}(T(V^*), C^*)$$

By the property of $(\ )^\circ$

$\gamma)$ $$\mathbf{Alg}(T(V^*), C^*) \cong \mathbf{Coalg}(C, T(V^*)^\circ),$$

hence by $\alpha), \beta), \gamma)$

$$\mathbf{Vect}(C, V^{**}) \cong \mathbf{Coalg}(C, T(V^*)^\circ)$$

mapping $f \in \mathbf{Vect}(C, V^{**})$ as follows through $\alpha), \beta), \gamma)$

$$f \overset{\alpha)}{\longmapsto} \left(V^* \hookrightarrow V^{***} \xrightarrow{f^*} C^*\right) \overset{\beta)}{\longmapsto} \left(\begin{matrix}\text{induced algebra map} \\ g : T(V^*) \to C^*\end{matrix}\right)$$

$$\overset{\gamma}{\longmapsto} \left(C \to C^{*\circ} \xrightarrow{g^\circ} T(V^*)^\circ\right)$$

The intermediate map g makes the diagram

$$\begin{tikzcd}
T(V^*) \arrow[drr, "g"] & & \\
V^* \arrow[u, hook] \arrow[r, hook] & V^{***} \arrow[r, "f^*"] & C^*
\end{tikzcd}$$

commute.

Dualizing and building further we obtain the commutative diagram

$$\begin{tikzcd}
T(V^*)^0 \arrow[d, hook] & & & \\
T(V^*)^* \arrow[d] & & & \\
V^{**} & V^{****} \arrow[l] & C^{**} \arrow[l, "f^{**}"'] \arrow[ull, bend right, "g^*"'] & C \arrow[l, hook'] \arrow[uulll, bend right, "g^0"']
\end{tikzcd} \qquad (\#)$$

Here "$\hookrightarrow$" are natural inclusions and

$C \longrightarrow C^{*o}$ is also a natural inclusion factoring

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr] & C^{**} \\ & C^{*o} \arrow[u, hook] \end{tikzcd}$$

The existence of such a factorization comes from the following

Lemma. The natural map $C \hookrightarrow C^{**}$ has its image in C^{*o}.

Proof. $c \in C \subseteq C^{**}$, $c^* \in C^*$

$\Rightarrow$ $c^* \triangleright c := c_{(1)} c^*(c_{(2)})$ is a left C^*-module structure.

Since $\Delta(c) = c_{(1)} \otimes c_{(2)}$ is a finite sum

$C^* \triangleright c$ is finite dimensional.

Now, to conclude that $c \in C^{*\circ}$ we need the

argument $2) \Rightarrow 1)$ as in the proof of the following

proposition, where for $f \in A^*$ $(a \triangleright f)(a') := f(a'a)$.

Proposition. TFAE

1) $f \in A^\circ$

2) $A \triangleright f$ finite dimensional.

Proof. Assume that $f \in A^\circ$. Then $m^*(f) \in A^\circ \otimes A^\circ \subset A^* \otimes A^*$ so $A \triangleright f$ finite dimensional since if $m^*(f) = f_{(1)} \otimes f_{(2)}$, $a \triangleright f = f_{(1)} \langle f_{(2)}, a \rangle$.

The latter holds since

$$(a \triangleright f)(a') = f(a'a) = m^*(f)(a' \otimes a) = \langle f_{(1)}, a' \rangle \langle f_{(2)}, a \rangle .$$

$\Rightarrow \quad A \triangleright f \in \operatorname{span}(f_{(1)}) \quad \Rightarrow \quad \dim(A \triangleright f) < \infty.$

Now, assume that $\dim(A \triangleright f) < \infty$.

then $I = \{a \in A \mid a \triangleright (A \triangleright f) = 0\}$ is an ideal, $\operatorname{codim} I < \infty$,

because it is the kernel of the algebra map

$\pi : A \longrightarrow \operatorname{End}(A \triangleright f) \quad , \quad a \longmapsto (a' \triangleright f \longmapsto a \triangleright (a' \triangleright f) = aa' \triangleright f)$

and $\dim \operatorname{End}(A \triangleright f) < \infty$.

But $f(I) = \{ f(a) \mid a \triangleright f = 0 \}$ and

$$f(a) = f(1 \cdot a) = (a \triangleright f)(1) = \varepsilon(a \triangleright f) = 0$$

$\Rightarrow \quad f \in A^{\circ}. \quad \blacksquare$

Now we come back to the proof of the Theorem. The left vertical composite in the diagram (#) is $\pi : T(V^*)^\circ \longrightarrow V^{**}$. The top diagonal composite is the coalgebra map $F : C \longrightarrow T(V^*)^\circ$ corresponding to the linear map $f : C \longrightarrow V^{**}$. One can check that the bottom horizontal composite is $f : C \longrightarrow V^{**}$. Thus we have the commutative diagram

$$
\begin{tikzcd}
T(V^*)^\circ \arrow[d, "\pi"'] & & C \arrow[ll, "F"'] \arrow[dll, phantom] \\
V^{**} & & C \arrow[ll, "f"]
\end{tikzcd}
$$

and $T(V^*)^\circ$ is the cofree coalgebra $C(V^{**})$. The final task is to show that there is some subcoalgebra in $C(V^{**})$ which is cofree on V, completing the construction of $C(V)$. We need the following

Lemma. Let V be a vector subspace in a vector space W. Then $C(V) = \sum_i C_i$, where C_i are subcoalgebras in $C(W)$ and that under $\pi : C(W) \to W$ $\pi(C_i) \subset V \subset W$.

Proof. Excercise. $\square$

Applying the latter Lemma to $W = V^{**}$ we finish the construction of $C(V)$. $\square$

Now we come back to the construction of $M(B,A)$. We start from the cofree coalgebra on $\mathbf{Vect}(B,A)$ $\pi : C(\mathbf{Vect}(B,A)) \longrightarrow \mathbf{Vect}(B,A)$ and consider the composite ρ

$$C(\mathbf{Vect}(B,A)) \otimes B \xrightarrow{\pi \otimes B} \mathbf{Vect}(B,A) \otimes B \xrightarrow{ev} A$$

Then we define $M(B,A) := \sum_i C_i$, C_i subcoalgebras in $C(\mathbf{Vect}(B,A))$ s.t. $\rho|_{C_i \otimes B} : C_i \otimes B \to A$

is a measuring. It is obvious that $\theta := \rho|_{M(B,A)} \otimes B$ is a measuring.

Now we prove the universal property of the thus obtained measuring. Suppose $\psi : C \otimes B \to A$ measures. By the universal property of $\mathbf{Vect}(-,-)$, there is a unique linear map $\lambda : C \to \mathbf{Vect}(B,A) \otimes B$ making the diagram

$$\begin{tikzcd} C \otimes B \arrow[rr, "\psi"] \arrow[dr, "\lambda \otimes B"'] & & A \\ & \mathbf{Vect}(B,A) \otimes B \arrow[ur, "ev"'] & \end{tikzcd} \qquad (*)$$

commute.

By the universal property of the cofree coalgebra $C(\mathbf{Vect}(B,A))$ there is a unique coalgebra map $F: C \longrightarrow C(\mathbf{Vect}(B,A))$ the diagram

$$\begin{tikzcd}
C \arrow[d, "F"'] \arrow[drr, "\lambda"] & & \\
C(\mathbf{Vect}(B,A)) \arrow[rr, "\pi"'] & & \mathbf{Vect}(B,A)
\end{tikzcd} \qquad (**)$$

Now we prove that the image of F is contained in $M(B,A)$ which defines the desired map into $M(B,A)$ and will prove the universal property for π.

From (*) and (**) we get a commutative diagram

$$\begin{tikzcd}
C \otimes B \arrow[rr, "\varphi"] \arrow[d, "F \otimes B"'] \arrow[drr, "\lambda \otimes B"] & & A \\
C(\mathbf{Vect}(B,A)) \otimes B \arrow[rr, "\pi \otimes B"'] & & \mathbf{Vect}(B,A) \otimes B \arrow[u, "ev"']
\end{tikzcd} \qquad (***)$$

To prove that the image of F is contained in $M(B,A)$ is the same as showing that the image of F measures B to A (it is a coalgebra since F is a coalgebra map).

The meaning to be $\mathrm{Im}\, F \otimes B \to A$
is the suitable restriction of $\rho = ev \circ (\pi \otimes B)$.

We have to check that for $D := \mathrm{Im}\, F$

1) $\rho(d \otimes b_1 b_2) = \rho(d_{(1)} \otimes b_1)\, \rho(d_{(2)} \otimes b_2)$

2) $\rho(d \otimes 1) = \varepsilon(d) 1$.

This follows from commutativity of $(***)$, the fact that F is a coalgebra map, and the measuring property for ψ.
Thus $(D, \rho|_D)$ measures B to A so by definition $D \subset M(B, A)$. Therefore the diagram

$$
\begin{tikzcd}
C \otimes B \arrow[d, "F \otimes B"'] \arrow[dr, "F \otimes A"] \arrow[drr, bend left, "\psi"] & & \\
& C(\mathbf{Vect}(B, A)) \otimes B \arrow[r, "ev \circ (\pi \otimes A)"] & A \\
M(B, A) \otimes B \arrow[ur, leftrightarrow] \arrow[urr, bend right, "\theta"'] & &
\end{tikzcd}
$$

Uniqueness of F comes from various universal mapping properties used to define it. $\square$

Exercises about $M(B,A)$.

Ex. 10. There are coalgebra maps

$$M(B,A) \otimes M(E,B) \longrightarrow M(E,A) \qquad \text{(composition)}$$

$$\mathbb{C} \longrightarrow M(A,A) \qquad \text{(units)}$$

satisfying associativity and unitality, which define an enrichment of the category of algebras in coalgebras.

Remark. Therefore $M(A,A)$ has the structure of a <u>bialgebra</u>, i.e. it is an algebra and a coalgebra, where the multiplication and the unit are coalgebra maps.

Ex. 11. Show that $M(A,A)$ is a coalgebra and an algebra such that the comultiplication and the counit are algebra maps.

Solution. Selfduality of commutative diagrams. $\square$

Ex. 12. $E(M(B,A)) = \mathbf{Alg}(B,A)$.

Ex. 13. Assume there is given a monoid map $M \xrightarrow{\rho} \mathbf{Alg}(A,A)$. Show that the monoid algebra $\mathbb{C}M$ has a bialgebra structure and ρ extends uniquely to a bialgebra map $\mathbb{C}M \to M(A,A)$.

Ex. 14. $\mathrm{Prim}(M(A,A)) = \mathrm{Der}(A)$

Ex. 15. Assume there is given a Lie algebra map $\mathfrak{g} \xrightarrow{\rho} \mathrm{Der}(A)$. Show that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ has a bialgebra structure and ρ extends uniquely to a bialgebra map
$$\mathcal{U}(\mathfrak{g}) \to M(A,A).$$

Ex 16. Construct the canonical coalgebra map

$$C \longrightarrow M(C^*, \mathbb{C}).$$

$$C^* \xrightarrow{\cong} \mathrm{Vect}(C, \mathbb{C})$$

Find the formula for the corresponding meaning $C \otimes C^* \longrightarrow \mathbb{C}$.

Solution.

$$c \otimes c^* \longmapsto c(c^*) := c^*(c)$$

$$c(c_1^* c_2^*) = c_{(1)}(c_1^*)\, c_{(2)}(c_2^*) = c_1^*(c_{(1)})\, c_2^*(c_{(2)}). \quad \square$$

Definition. Let D be a coalgebra and B be an algebra. Then
$$B(D) := T(B \otimes D) \big/ \left(b_1 b_2 \otimes d - (b_1 \otimes d_{(1)}) \otimes (b_2 \otimes d_{(2)}) \right)$$
is a quotient algebra of the free algebra $T(B \otimes D)$ generated by the vector space $B \otimes D$.

Remark. $B(D)$ is generated by symbols $b(d)$ bilinear in b and d subject to the relations
$$(b_1 b_2)(d) = b_1(d_{(1)})\, b_2(d_{(2)}) .$$

Theorem. $M(B, \mathrm{Vect}(D,A)) \simeq M(B(D), A)$ iso of coalgebras natural in A, B and D.

Proof. $\mathrm{Coalg}(C, M(B, \mathrm{Vect}(D,A))) = \mathrm{Meas}^C(B, \mathrm{Vect}(D,A)) \ni f$

$f(c \otimes b) = c(b) \in \mathrm{Vect}(D,A)$.

$$c(b_1 b_2)(d) = (c_{(1)}(b_1)\, c_{(2)}(b_2))(d) = (c_{(1)}(b_1))(d_{(1)}) \cdot (c_{(2)}(b_2))(d_{(2)})$$

$$\mathrm{Coalg}(C, M(B(D), A)) = \mathrm{Meas}^C(B(D), A)$$

Try $\mathrm{Meas}^C(B, \mathrm{Vect}(D,A)) \longrightarrow \mathrm{Meas}^C(B(D), A)$ (*)

$$(c \otimes b \mapsto (d \mapsto c(b)(d))) \longmapsto (c \otimes b(d) \longmapsto c(b)(d))$$

Does it define well a desired map? It is clear that then it must be a bijection.

Let's check:

$c(b_1(d_1)\dots b_n(d_n)) := c_{(1)}(b_1)(d_1)\dots c_{(n)}(b_n)(d_n)$ measuring

well defined on the level of the free algebra generated by symbols $b(d)$. Compatibility with relations defining $B(D)$:

$$c((b_1 b_2)(d) - b_1(d_{(1)})\, b_2(d_{(2)}))$$

$$= c(b_1 b_2)(d) - c_{(1)}(b_1)(d_{(1)}) \cdot c_{(2)}(b_2)(d_{(2)})$$

$$= (c_{(1)}(b_1) \cdot c_{(2)}(b_2))(d) - c_{(1)}(b_1)(d_{(1)}) \cdot c_{(2)}(b_2)(d_{(2)})$$

$$= c_{(1)}(b_1)(d_{(1)}) \cdot c_{(2)}(b_2)(d_{(2)}) - c_{(1)}(b_1)(d_{(1)}) \cdot c_2(b_2)(d_{(2)}) = 0$$

$\Rightarrow$ $c(b(d)) := c(b)(d)$ extends uniquely to a measuring $C \otimes B(D) \longrightarrow A$. $\square$

Remark. After passing to the opposite category of algebras, enriched in coalgebras this is an analog of the adjunction

$$\mathbf{Space}(S\times X, Y) = \mathbf{Space}(X, Y^S)$$

where $S\times X$ is the disjoint union $\coprod_{s\in S} \{s\}\times X$ of copies of a space X parameterized by a set S, and Y^S is the space of maps from the set S to the space Y, according to the following dictionary

Space	**Alg**^{op}
spaces X, Y	algebras A, B
a set S	a coalgebra D
a space $S \times X$	an algebra $\mathbf{Vect}(D, A)$
a space Y^S	an algebra $B(D)$
a set $\mathbf{Space}(S \times X, Y)$	a coalgebra $M(B, \mathbf{Vect}(D, A))$
a set $\mathbf{Space}(X, Y^S)$	a coalgebra $M(B(D), A)$